

REGULARITY OF NONLINEAR GENERALIZED FUNCTIONS: A COUNTEREXAMPLE IN THE NONSTANDARD SETTING

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ABSTRACT. Regularity theory in generalized function algebras of Colombeau type is largely based on the notion of \mathcal{G}^∞ -regularity, which reduces to \mathcal{C}^∞ -regularity when restricted to Schwartz distributions. Surprisingly, in the nonstandard version of the Colombeau algebras, this basic property of \mathcal{G}^∞ -regularity does not hold.

1. INTRODUCTION

Generalized function algebras are differential algebras that contain (up to isomorphism) the space of Schwartz distributions as a differential subspace, and in which the product of \mathcal{C}^∞ -functions coincides with their usual product. They have been introduced by J. F. Colombeau [1] and find their main applications in the study of nonlinear PDE (e.g. in General Relativity [7, 10, 16]) and PDE with highly singular data or coefficients (e.g. [3, 6, 9, 12]) for which the distributional solution concept does not make sense.

A well-developed qualitative theory of generalized solutions to PDE has emerged based on the notion of \mathcal{G}^∞ -regularity [12, 15] and the corresponding \mathcal{G}^∞ -microlocal regularity, aiming at describing the propagation of singularities of PDE (e.g. [2, 4, 5, 8, 20]). The basic property which makes \mathcal{G}^∞ -regularity of a generalized function a suitable concept to study its regularity is that the \mathcal{G}^∞ -regular distributions (viewed as elements of the generalized function algebra) are exactly the \mathcal{C}^∞ -functions.

Due to inherent similarities in the construction of the Colombeau algebras with the construction of algebras of functions in nonstandard models of analysis [14], also a nonstandard version of the Colombeau algebras has been constructed [13]. This variant enjoys similar, but in some aspects nicer properties than the standard algebra. E.g., the ring of (real) scalars in the nonstandard algebra is a (totally) ordered field (and not, as in the standard algebra, a partially ordered ring with zero divisors), and a Hahn-Banach extension property for continuous linear functionals holds [17] (which fails in the standard algebra [18]). More generally, the full principles of nonstandard analysis (such as the Transfer Principle) are available for representatives of generalized functions in the nonstandard algebra (whereas only a restricted version holds in the standard algebras [19]). The above-mentioned basic property of \mathcal{G}^∞ -regularity was therefore expected to hold also in the nonstandard version. However, the proof remained elusive. The goal of this paper is to construct an explicit counterexample.

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2. PRELIMINARIES

Let $\Omega \subseteq \mathbb{R}^d$ be open. We work in the so-called special Colombeau algebra $\mathcal{G}(\Omega) = \mathcal{M}(\Omega)/\mathcal{N}(\Omega)$ [7, §1.2], where

$$\begin{aligned}\mathcal{M}(\Omega) &= \{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(\Omega)^{(0,1]} : \\ &\quad (\forall K \subset\subset \Omega)(\forall \alpha \in \mathbb{N}^d)(\exists N \in \mathbb{N})(\exists \varepsilon_0 \in (0, 1])(\forall \varepsilon \in (0, \varepsilon_0))(\sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| \leq \varepsilon^{-N})\} \\ \mathcal{N}(\Omega) &= \{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(\Omega)^{(0,1]} : \\ &\quad (\forall K \subset\subset \Omega)(\forall \alpha \in \mathbb{N}^d)(\forall m \in \mathbb{N})(\exists \varepsilon_0 \in (0, 1])(\forall \varepsilon \in (0, \varepsilon_0))(\sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| \leq \varepsilon^m)\}.\end{aligned}$$

We denote by $[u_\varepsilon] \in \mathcal{G}(\Omega)$ the generalized function with representative $(u_\varepsilon)_\varepsilon$. $\mathcal{G}^\infty(\Omega)$ is the subalgebra of those $u \in \mathcal{G}(\Omega)$ with a representative $(u_\varepsilon)_\varepsilon$ satisfying

$$(\forall K \subset\subset \Omega)(\exists N \in \mathbb{N})(\forall \alpha \in \mathbb{N}^d)(\exists \varepsilon_0 \in (0, 1])(\forall \varepsilon \in (0, \varepsilon_0))(\sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| \leq \varepsilon^{-N}).$$

Let \mathcal{U} be a free ultrafilter on $(0, 1]$ with $(0, \delta] \in \mathcal{U}$ for each $\delta > 0$. The nonstandard version of the special algebra is ${}^\rho\mathcal{E}(\Omega) = \mathcal{M}(\Omega)/\mathcal{N}(\Omega)$ [11, 13], where

$$\begin{aligned}\mathcal{M}(\Omega) &= \{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(\Omega)^{(0,1]} : \\ &\quad (\forall K \subset\subset \Omega)(\forall \alpha \in \mathbb{N}^d)(\exists N \in \mathbb{N})(\exists S \in \mathcal{U})(\forall \varepsilon \in S)(\sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| \leq \varepsilon^{-N})\} \\ \mathcal{N}(\Omega) &= \{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(\Omega)^{(0,1]} : \\ &\quad (\forall K \subset\subset \Omega)(\forall \alpha \in \mathbb{N}^d)(\forall m \in \mathbb{N})(\exists S \in \mathcal{U})(\forall \varepsilon \in S)(\sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| \leq \varepsilon^m)\}\end{aligned}$$

(to be precise, this is the nonstandard version of the algebra in [13] in the case where the nonstandard model is constructed using the ultrafilter \mathcal{U} on $(0, 1]$ and where the fixed positive infinitesimal $\rho \in {}^*\mathbb{R}$ has representative $(\varepsilon)_{\varepsilon \in (0, 1]}$).

${}^\rho\mathcal{E}^\infty(\Omega)$ is the subalgebra of those $u \in {}^\rho\mathcal{E}(\Omega)$ with a representative $(u_\varepsilon)_\varepsilon$ satisfying

$$(\forall K \subset\subset \Omega)(\exists N \in \mathbb{N})(\forall \alpha \in \mathbb{N}^d)(\exists S \in \mathcal{U})(\forall \varepsilon \in S)(\sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| \leq \varepsilon^{-N}).$$

Let ϕ be a function in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ which satisfies the moment conditions $\int_{\mathbb{R}^d} \phi(x) dx = 1$ and $\int_{\mathbb{R}^d} x^\alpha \phi(x) dx = 0$ for each $\alpha \in \mathbb{N}^d \setminus \{0\}$, and denote $\phi_\varepsilon(x) := \frac{1}{\varepsilon^d} \phi(x/\varepsilon)$. Then the embedding ι of the space $\mathcal{D}'(\Omega)$ of Schwartz distributions into the Colombeau algebra $\mathcal{G}(\Omega)$ is given, for $T \in \mathcal{E}'(\Omega)$, by $\iota(T) = [T * \phi_\varepsilon]$ [7, §1.2].

Our counterexample will be based on a careful analysis of a proof of the result for the standard algebra. For the sake of the readability of the counterexample, we recall the proof in an elementary version:

Theorem. [12, 15] $\mathcal{G}^\infty(\Omega) \cap \iota(\mathcal{D}'(\Omega)) = \mathcal{C}^\infty(\Omega)$.

Proof. 1. Let $T \in \mathcal{E}'(\Omega)$ with $\iota(T) = [T * \phi_\varepsilon] \in \mathcal{G}^\infty(\Omega)$. Then the Fourier transform $\widehat{T} \in \mathcal{C}^\infty(\Omega) \cap \mathcal{S}'(\mathbb{R}^d)$. Because $\iota(T) \in \mathcal{G}_c^\infty(\Omega)$, we have

$$(\exists N \in \mathbb{N})(\forall \alpha \in \mathbb{N}^d)(\exists \varepsilon_0 > 0)(\forall \varepsilon \leq \varepsilon_0)(\sup_{x \in \mathbb{R}^d} \langle x \rangle^{2d} |\partial^\alpha (T * \phi_\varepsilon)(x)| \leq \varepsilon^{-N}).$$

where $\langle x \rangle^2 = 1 + |x|^2$. For the Fourier transform $\widehat{\iota(T)} = [\widehat{T} \cdot \widehat{\phi}(\varepsilon \cdot)]$, we have

$$|\xi^\alpha \widehat{T}(\xi) \widehat{\phi}(\varepsilon \xi)| = |\mathcal{F}(\partial^\alpha (T * \phi_\varepsilon))(\xi)| \leq \int_{\mathbb{R}^d} |\partial^\alpha (T * \phi_\varepsilon)(x)| dx.$$

Hence

$$(1) \quad (\exists N \in \mathbb{N})(\forall \alpha \in \mathbb{N}^d)(\exists \varepsilon_0 > 0)(\forall \varepsilon \leq \varepsilon_0)(\forall \xi \in \mathbb{R}^d)(|\xi^\alpha \widehat{T}(\xi) \widehat{\phi}(\varepsilon \xi)| \leq \varepsilon^{-N}).$$

From this, we show that $\widehat{T} \in \mathcal{S}(\mathbb{R}^d)$. Since $\widehat{\phi}(0) = \int_{\mathbb{R}^d} \phi = 1$, there exists $\delta > 0$ such that $|\widehat{\phi}(\eta)| \geq 1/2$ for $|\eta| \leq \delta$. For $\xi \in \mathbb{R}^d$ with $|\xi| \geq \delta/\varepsilon_0$, choosing $\varepsilon = \delta/|\xi|$ in (1) (hence $\varepsilon \leq \varepsilon_0$) yields

$$(2) \quad (\exists N \in \mathbb{N})(\forall \alpha \in \mathbb{N}^d)(\forall \xi \in \mathbb{R}^d, |\xi| \geq \delta/\varepsilon_0)(|\xi^\alpha \widehat{T}(\xi)| \leq 2|\xi|^N/\delta^N).$$

We proceed similarly for all derivatives. Hence $\widehat{T} \in \mathcal{S}(\mathbb{R}^d)$, and $T \in \mathcal{S}(\mathbb{R}^d) \subseteq \mathcal{C}^\infty(\mathbb{R}^d)$.

2. If $T \in \mathcal{D}'(\Omega)$, then we can use a cut-off to reduce to the case $T \in \mathcal{E}'(\Omega)$. \square

3. COUNTEREXAMPLE IN ${}^\rho\mathcal{E}(\Omega)$

We give a counterexample in the nonstandard algebra ${}^\rho\mathcal{E}(\Omega)$ for the case $\Omega = \mathbb{R}$. Essentially, only the step (1) \Rightarrow (2) in the proof of the previous section fails in this setting, which we will exploit. We use an embedding ι which is given, for $T \in \mathcal{E}'(\mathbb{R})$, by $\iota(T) = [T * \phi_\varepsilon]$, where the mollifier $\phi \in {}^*\mathcal{C}^\infty(\mathbb{R})$ satisfies $|\widehat{\phi}(\xi)| \leq C_p \langle \xi \rangle^{-p}$ for each $p \in \mathbb{N}$ (for some $C_p \in \mathbb{R}$; in particular, this includes the case where $\phi \in \mathcal{S}(\mathbb{R})$; the proof can easily be extended if the inequality only holds for $C_p = |\ln \rho| = \lfloor |\ln \varepsilon| \rfloor \in {}^*\mathbb{R}$, thus including also the case where ϕ is given by a mollifier as in [13, Lemma 3.1]). As usual, $\phi_\varepsilon(x) := \varepsilon^{-1} \phi(x/\varepsilon)$.

Example. Since \mathcal{U} is an ultrafilter, $S \in \mathcal{U}$ or $S^c \in \mathcal{U}$ for any $S \subseteq (0, 1]$. Let $\varepsilon_n := 1/2^{(n^n)}$. This sequence has the property that $(\forall p \in \mathbb{N})(\exists N)(\forall n \geq N)(\varepsilon_{n+1} \leq \varepsilon_n^p)$. Consider $S := \bigcup_{n \in \mathbb{N}} (\varepsilon_{6n+3}, \varepsilon_{6n}]$. We consider first the case that $S \in \mathcal{U}$. Then let

$$(3) \quad \widehat{T}(\xi) := \sum_{m=1}^{\infty} \psi(\xi - \varepsilon_{6m-1}^{-1}),$$

where $\psi \in \mathcal{S}(\mathbb{R})$ with $\widehat{\psi} \in \mathcal{D}(\mathbb{R})$.

We claim that $T \in \mathcal{E}'(\mathbb{R}) \setminus \mathcal{C}^\infty(\mathbb{R})$, but $\iota(T) = [T * \phi_\varepsilon] \in {}^\rho\mathcal{E}^\infty(\mathbb{R})$.

Proof. With the usual estimates (e.g. Peetre's inequality), one sees that the sum (3) converges uniformly on compact subsets and that \widehat{T} is a bounded \mathcal{C}^∞ -function. The sum thus converges in $\mathcal{S}'(\mathbb{R})$, and $T \in \mathcal{S}'(\mathbb{R})$ is well-defined. As

$$\langle T, \varphi \rangle = \sum_{m=1}^{\infty} \int_{\mathbb{R}} \mathcal{F}^{-1}(\psi)(x) e^{i\varepsilon_{6m-1}^{-1}x} \varphi(x) dx, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}),$$

$\text{supp } T \subseteq \text{supp}(\mathcal{F}^{-1}(\psi))$, hence $T \in \mathcal{E}'(\mathbb{R})$. Seeking a contradiction, suppose that $T \in \mathcal{C}^\infty(\mathbb{R})$. Then $T \in \mathcal{S}(\mathbb{R})$, hence $\widehat{T} \in \mathcal{S}(\mathbb{R})$, too, but due to its definition, $\widehat{T}(\xi) \not\rightarrow 0$ as $\xi \rightarrow +\infty$. It remains to be shown that $\iota(T) \in {}^\rho\mathcal{E}^\infty(\mathbb{R})$. We first show that $\widehat{\iota(T)}$ satisfies the analogue of equation (1). Let $p \in \mathbb{N}$ arbitrary ($p > 0$) and $\varepsilon \in (\varepsilon_{6n+3}, \varepsilon_{6n}]$. We proceed to show that for n sufficiently large,

$$(4) \quad \langle \xi \rangle^p |\widehat{T * \phi_\varepsilon}(\xi)| = \langle \xi \rangle^p |\widehat{T}(\xi) \widehat{\phi}(\varepsilon\xi)| \leq \sum_{m=1}^{\infty} \langle \xi \rangle^p |\psi(\xi - \varepsilon_{6m-1}^{-1}) \widehat{\phi}(\varepsilon\xi)| \leq \varepsilon^{-1}.$$

As $\psi \in \mathcal{S}(\mathbb{R})$, we find $C_p \in \mathbb{R}$ such that (by Peetre's inequality)

$$\sum_{m=1}^n \langle \xi \rangle^p |\psi(\xi - \varepsilon_{6m-1}^{-1})| \leq C_p \sum_{m=1}^n \langle \xi \rangle^p \langle \xi - \varepsilon_{6m-1}^{-1} \rangle^{-p} \leq C_p' \sum_{m=1}^n \varepsilon_{6m-1}^{-p} \leq C_p'' \varepsilon_{6n-1}^{-p},$$

which is at most $\varepsilon_{6n}^{-1} \leq \varepsilon^{-1}/2$, as soon as n is sufficiently large.

Further, if $|\xi| \leq \varepsilon_{6n+4}^{-1}$ and $m > n$, then

$$|\psi(\xi - \varepsilon_{6m-1}^{-1})| \leq C \langle \xi - \varepsilon_{6m-1}^{-1} \rangle^{-1} \leq C \langle \varepsilon_{6m-1}^{-1} - \varepsilon_{6n+4}^{-1} \rangle^{-1} \leq 2C \varepsilon_{6m-1}$$

and thus

$$\sum_{m=n+1}^{\infty} \langle \xi \rangle^p |\psi(\xi - \varepsilon_{6m-1}^{-1}) \widehat{\phi}(\varepsilon\xi)| \leq C_p \varepsilon_{6n+4}^{-p} \sum_{m=n+1}^{\infty} \varepsilon_{6m-1} \leq 2C_p \varepsilon_{6n+4}^{-p} \varepsilon_{6n+5} \leq 1$$

as soon as n is sufficiently large. On the other hand, if $|\xi| \geq \varepsilon_{6n+4}^{-1}$, then

$$|\widehat{\phi}(\varepsilon\xi)| \leq C_p \langle \varepsilon\xi \rangle^{-p-1} \leq C_p' \varepsilon^{-p-1} \langle \xi \rangle^{-p-1} \leq C_p'' \varepsilon_{6n+3}^{-p-1} \langle \xi \rangle^{-p} \varepsilon_{6n+4} \leq \langle \xi \rangle^{-p}$$

as soon as n is sufficiently large, and thus

$$\sum_{m=n+1}^{\infty} \langle \xi \rangle^p |\psi(\xi - \varepsilon_{6m-1}^{-1}) \widehat{\phi}(\varepsilon\xi)| \leq \sum_{m=n+1}^{\infty} |\psi(\xi - \varepsilon_{6m-1}^{-1})| \leq C.$$

This proves (4), which implies, still for $\varepsilon \in (\varepsilon_{6n+3}, \varepsilon_{6n}]$ and n sufficiently large,

$$|D^k(T * \phi_\varepsilon)(x)| = |\mathcal{F}^{-1}(\xi^k \cdot \widehat{T * \phi_\varepsilon})(x)| \leq \int_{\mathbb{R}} \langle \xi \rangle^k |\widehat{T * \phi_\varepsilon}(\xi)| d\xi \leq \varepsilon^{-1} \int_{\mathbb{R}} \frac{d\xi}{\langle \xi \rangle^2},$$

hence $\iota(T) \in {}^\rho\mathcal{E}^\infty(\mathbb{R})$. □

In the case when $S^c = \bigcup_{n \in \mathbb{N}} (\varepsilon_{6n}, \varepsilon_{6n-3}] \in \mathcal{U}$, we similarly consider $\widehat{T}(\xi) := \sum_{m=0}^{\infty} \psi(\xi - \varepsilon_{6m+2}^{-1})$.

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